1 Nash Equilibrium, Prisoner Dilemma, and Collusion

Consider the Bertrand model with two possible prices $p_i \in \{p^m, p\}$, for $p \in [c, p^m)$. Recall that, in this context, a **Nash Equilibrium** is a profile of prices (p_1^*, p_2^*) such that:

$$\Pi^1(p_1^*, p_2^*) \geq \Pi^1(p_1, p_2^*) \quad \text{for } p_1 \in \{p^m, p\},$$
$$\Pi^2(p_1^*, p_2^*) \geq \Pi^2(p_1^*, p_2) \quad \text{for } p_2 \in \{p^m, p\}.$$

a) Solve all Nash Equilibria of the game below depending on the value of p (expressed in terms of $\Pi(p)$).

	Firm 2	
Firm 1	p^m	p
p^m	$\frac{\Pi(p^m)}{2}, \frac{\Pi(p^m)}{2}$	$0, \Pi(p)$
p	$\Pi(p), 0$	$\frac{\Pi(p)}{2}, \frac{\Pi(p)}{2}$

Hint: treat $p = c$ as a special case.

Also, we say that an action profile (p_1^{PO}, p_2^{PO}) is Pareto Optimal (or Pareto Efficient) if no other profile (p_1, p_2) exists such that at least one firm can increase its profits without reducing the profits of the other firm. (It is impossible to make one firm better off without making the other firm worse off.) That is, there is no other profile (p_1, p_2) for which:

$$\Pi^1(p_1, p_2) \geq \Pi^1(p_1^{PO}, p_2^{PO}),$$
$$\Pi^2(p_1, p_2) \geq \Pi^2(p_1^{PO}, p_2^{PO}),$$

with at least one of the inequalities being strict.

b) Identify the profiles of prices in a) that are Pareto optimal.

c) For what values of p does the above game represent a Prisoner's Dilemma? Why?

d) Now suppose that both firms set prices as continuous variables: $p_i \in \mathbb{R}_+$, for $i = 1, 2$ (i.e., there are more than two price strategies of each firm). What are the similarities between the Bertrand game and the Prisoner's Dilemma?

e) Now suppose that $p_i \in \mathbb{R}_+$ for $i = 1, 2$, and that the game is repeated $T < \infty$ times (supergame framework). Suppose that firms take into account discounted profits with discount factor δ. Is it possible for firms to sustain a collusive outcome? If so under what conditions?

f) Same question as before but assuming infinite repetitions.

2 Collusion

Consider two identical firms with constant marginal costs, c, and no capacity constraints who both discount future payoffs with discount factor δ. They interact repeatedly in the same market, using Cournot competition. Demand is given by

$$P(Q) = 1 - 2Q, \quad \text{where } Q = q_1 + q_2.$$

a) Suppose the two firms successfully sustain a collusive game where each sets quantity so that price is at the monopoly level in every period. They share the monopoly profits equally. This strategy continues forever. Calculate each firm's total discounted profits.

b) Now suppose that one firm deviates in a single period. What is the deviating firm's choice of quantity and what is its profit in that period?

c) Assume that firms follow a "trigger strategy" profile: if someone has deviated, all firms set static Cournot quantities forever. Show that collusion is sustainable provided firms are patient enough.

d) Now suppose that the punishment (all firms setting Cournot quantities) lasts for only T periods, with $T < \infty$. Write down an inequality that must hold for any T that is large enough to deter deviation. Simplify the expression as much as you can.

e) Give two examples of situations (market conditions or types of firms) where collusion may be difficult to sustain.

3 Collusion in a Cournot Oligopoly

Consider an n-firm supergame framework where each stage follows the Cournot model. $\delta \in (0, 1)$ is the discount factor. The inverse-demand function is

$$P(Q) = a - Q,$$

where Q represents the total quantity. The marginal cost of production is constant and equal to c, and we assume that $a > c$. Firms are considering a collusive agreement to split among each other the monopoly profit instead of being in the equilibrium of the Cournot model.

a) Calculate the limit discount factor beyond which collusion is sustainable.

b) How does the limit discount factor vary with the number of firms in the market?

Hint: if a firm deviates from the agreement, it chooses quantity to maximize its profit in one period, assuming other firms are choosing the collusion quantity.